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Existence of edge waves along three-dimensional periodic structures

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Existence of edge waves travelling along three-dimensional periodic structures is considered within the linear water-wave theory. A condition ensuring the existence is derived by analysing the spectrum of a suitably defined trace operator. The sufficient condition is a simple inequality comparing a weighted volume integral, taken over the submerged part of an element in the infinite array of identical obstacles, to the area of the free surface pierced by the obstacle. Various examples are given, and the results are extended to edge waves along periodic coastlines and over a periodically varying ocean floor.

Key words: edge waves, periodic structures, sufficient condition, trace operator, trapped modes

1. Introduction

In this paper, we are concerned with the problem of interaction of linear water waves with three-dimensional periodic structures, that is, arrays of obstacles or protrusions from coastline or ocean floor. More precisely, we are interested in proving the existence of localized solutions, also known as trapped modes, which are waves guided along the periodic structures, *e.g.* an array of identical obstacles, an underwater ridge or a periodic coastline, but decay at large distances from them, that is, they are trapped in their vicinity. Stokes (1846) seems to have been the first to come up with a trapped mode while presenting an analytical solution describing a wave travelling along a uniformly sloping beach. Much later, Ursell (1952) generalized this result showing, in particular, that Stokes' solution is only the first in a finite family of trapped modes. See also Roseau (1958) for further extensions and for a rigorous derivation of Ursell's formulae. These oscillating solutions guided along, and confined to the vicinity of, coastlines are called edge waves. Trapped edge waves have also been shown to exist along infinite horizontal submerged cylinders by Ursell (1951) (see also Ursell 1987), underwater mountain ridges by Jones (1953) and Garipov (1967) and along coastlines with a non-uniformly sloping beach (cf. Bonnet-Ben Dhia & Joly 1993).

For all the edge waves mentioned above, the solutions we are looking for are generated by periodic structures. However, in contrast to the previous works, our

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problem is truly three-dimensional since the obstacles do not need to have a constant horizontal cross-section, that is, they can be of any (identical) shape, and the (non-straight) periodic coastline does not need to have a vertical cliff face. We do assume though that the ocean floor is either flat or varying periodically in the same direction and with the same period as the array of obstacles.

In Evans & Linton (1993) and Evans & Fernyhough (1995), edge waves were found numerically along a periodic coastline when the protrusions were thin parallel vertical plates or rectangular blocks. Similarly, in McIver, Linton & McIver (1998) and Porter & Evans (1999), trapped modes were computed, again numerically, in the vicinity of an infinite array of vertical cylinders extending uniformly throughout the depth. In Linton & McIver (2002), edge waves were shown to exist along general periodic coastlines and infinite arrays of obstacles, but still only when the ocean has constant depth and the obstacles have a uniform cross-section. All these problems can be treated as two-dimensional since the z-dependence can be factored out right at the outset due to the vertical uniformity. Incidentally, these trapped edge waves are also called Rayleigh–Bloch surface waves since the resulting two-dimensional problem is closely related to the problem of existence of Rayleigh–Bloch waves along acoustic or electromagnetic diffraction gratings (cf. Wilcox 1984). For the existence of acoustic edge waves along a periodic array of obstacles, we also refer to Sukhinin (1998), for similar results and reasoning as in Linton & McIver (2002), and to Kamotskii & Nazarov (1999), for a more general approach (cf. § 6.4).

There are practically no results towards existence of edge waves along periodic structures that do not have a uniform cross-section in at least one direction. We point out the work by Porter & Porter (2001) where the authors consider trapping of water waves by a periodically varying ocean bed. Using the so-called mild-slope approximation on the bottom topography, they numerically justify the existence of trapped modes provided the periodic part of the bottom is elevated, on average, sufficiently high above the mean level of the ocean floor.

A trapped edge wave propagating along a periodic structure corresponds to a non-trivial solution (eigenfunction) of a certain spectral boundary-value problem arising from the linear water-wave theory (cf. Kuznetsov, Maz'ya & Vainberg 2002). In fact, given the periodicity of the structure, say $l > 0$ in the y-coordinate direction, it is sufficient to solve the spectral problem in an infinite-periodicity cell of width $l > 0$ with certain (artificial) boundary conditions at the (artificial) lateral walls of the cell related to the expected propagation of the edge wave in the y-direction. The trapped-wave solution is associated with an eigenvalue below a (positive) lower bound of the essential spectrum of the spectral problem. There may also exist trapped modes which are embedded into the essential spectrum; see e.g. the review article by Linton & McIver (2007). In connection with edge waves generated by periodic structures, embedded trapped modes can be found, as observed in Evans, Levitin & Vassilev (1994), if Neumann boundary conditions are imposed at the lateral walls and the obstacle is assumed to be symmetric with respect to the centreline $y = l/2$ of the periodicity cell. The resulting solutions correspond to standing waves.

In the next section, we introduce our notation and assumptions leading to the mathematical setting of the problem as a spectral boundary-value problem in a periodicity cell. In § 3, following Nazarov (2008, 2009*b*) (see also Nazarov & Videman 2009), we introduce, in a suitable Hilbert space, a trace operator T_β associated with the standard variational formulation of the spectral problem in the periodicity cell. This operator is continuous, positive and self-adjoint, and its essential spectrum covers the segment $[0, \mu_\beta^\dagger]$, where $\mu_\beta^\dagger = (\lambda_\beta^\dagger)^{-1}$, and $\lambda_\beta^\dagger > 0$ is the positive lower bound of the continuous spectrum of the spectral boundary-value problem. The cutoff eigenvalue $\lambda_\beta^\dagger$, and the associated non-trivial eigenfunction $\phi_\beta^\dagger$, can be determined explicitly by solving a model spectral problem in the infinite rectangular prism corresponding to the periodicity cell in the absence of obstacles.

In § 4, using a particular test function to estimate the norm of the operator T_β from below, we derive a condition which guarantees that the discrete spectrum of T_β is non-empty and, consequently, secures the existence of a trapped mode. This condition is a simple inequality comparing a certain weighted volume integral taken over the size of the obstacle to the surface integral (area) of the horizontal cross-section sliced from the obstacle by the free surface. Our approach resembles the variational method formulated by Kamotskii & Nazarov (1999) (see also Kamotskii & Nazarov 2003) for similar spectral problems in the elasticity and acoustics. However, given that in the water-wave problems, the spectral parameter appears in the boundary conditions, it is essential to derive a suitable trace operator formulation for the variational method to work. In Nazarov (2009*b*,*d*), this approach was applied to two-dimensional water-wave problems in fluid domains of infinite depth. In Nazarov (2009*d*) also, the three-dimensional channel problem was addressed with the usual symmetry assumptions and artificial boundary conditions (see Evans *et al.* 1994) and, in Nazarov (2009*a*), conditions guaranteeing the existence of trapped modes were derived in the (essentially) two-dimensional case where an infinite horizontal cylinder is placed over a periodic bottom, with the axis of the cylinder perpendicular to the direction of the bottom periodicity. In Nazarov & Videman (2009), the authors established conditions for the existence of edge waves along infinite horizontal cylinders in a two-layer fluid.

In § 5, we consider the case where the trapped edge waves, if they exist, are non-propagating. Since this situation leads to a continuous spectrum starting from zero, we assume, following the reasoning of Evans *et al.* (1994), that the obstacle is symmetric with respect to the centreline $y = l/2$ and introduce a new spectral problem in the left-half of the periodicity cell with homogeneous Neumann and Dirichlet boundary conditions at the lateral walls at $y = 0$ and $y = l/2$, respectively. The continuous spectrum of the new problem has a positive lower bound which allows us to derive a sufficient condition for the existence of a trapped mode, again in the form of an inequality between weighted volume and surface integrals, taken over the same sets as above but with different weight functions, depending, in particular, on the y-coordinate. After even and odd extensions through the lateral boundaries, the resulting trapped-mode solution is shown to correspond to a $2l$-periodic standing wave.

In §§ 6 and 7, we will present several examples satisfying the sufficient condition and, therefore, producing trapped edge waves, such as any (periodic array of) submerged obstacles, surface-piercing columns with uniform and non-uniform cross-sections and barriers with apertures. We will also prove the non-necessity of the sufficient condition and, further, give a simple proof of the comparison principle.

Finally, in § 8, we extend our results to periodically varying coastlines and periodically varying ocean floor topographies. In the former case, sufficient conditions can be derived in much the same way as in §§ 4 and 5; the integrals over the obstacle are replaced with weighted integrals over and along the upper boundary of, the cross-section of the periodicity cell, with the positive weight function $H = H(y, z)$ corresponding to the coastline topography. On the other hand, if the bottom topography is given by a periodic function, we cannot explicitly solve the spectral model problem. The cutoff value of the continuous spectrum is, however, still positive,

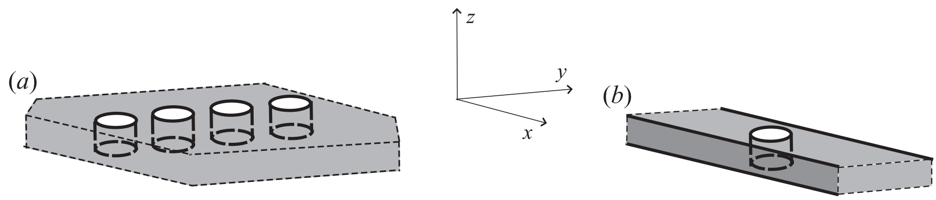

FIGURE 1. An infinite array (a) and the periodicity cell (b).

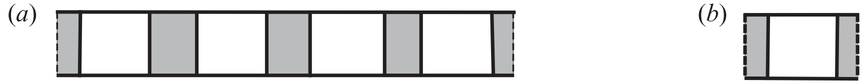

FIGURE 2. Cross-sections in the (y,z)-plane of the pictures in figure 1.

and we can derive a sufficient condition for the existence of a trapped mode. This condition is an inequality comparing the Dirichlet integral of the eigenfunction associated with the cutoff value, taken over the volume of the obstacle, to the square integral of the eigenfunction calculated over the part of the free surface pierced by the obstacle. In both cases, we will give some examples leading to the existence of trapped edge waves.

2. Formulation of the problem in a domain of constant depth

Consider an infinite layer of an incompressible, homogeneous, inviscid liquid lying on a flat bottom. Cartesian coordinates are chosen so that the (x, y)-plane coincides with the mean position of the free surface separating the fluid layer from another fluid (air) of negligible density and the z-coordinate points upwards.

Assume that the layer contains an infinite array of rigid obstacles which are aligned along the y-axis and may extend throughout the finite depth $(-h, 0)$; see *e.g.* the array of vertical columns in figure 1(a) and the cross-section in the (y, z)-plane of the same picture in figure 2(a). In what follows, we will mainly draw these two-dimensional (cross-sectional) pictures, as in figure 2, of the three-dimensional objects. Moreover, assume that the (groups of) obstacles are identical so that the fluid layer can be divided into an infinite family of identical cells, each of them extending to infinity in the x-direction, having finite width $l > 0$ in the y-direction and containing the same obstacles.

We denote the fluid layer by $\varXi = \mathbb{R}^2 \times (-h, 0)$, and let $\Pi \subset \varXi$ be the prismatic periodicity cell without obstacles, i.e.

$$\Pi = \{(x, y, z) \in \varXi \; : \; y \in (0, l)\}, \tag{2.1}$$

and assume that $\varTheta \subset \Pi$ is a bounded, open, non-empty, not necessarily connected set corresponding to a certain member in the family of identical obstacles (or groups of them). Given the l-periodicity in y, we define the sets Π_j and $\varTheta_j$ by

$$\left.\begin{aligned} \Pi_j &= \{(x, y, z) \in \varXi \; : \; y \in (jl, (j+1)l)\}, \quad j \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}, \\ \varTheta_j &= \{(x, y, z) \in \Pi_j \; : \; (x, y - lj, z) \in \varTheta\}, \quad j \in \mathbb{Z}. \end{aligned}\right\} \tag{2.2}$$

We assume that the fluid region

$$\Omega = \Xi \setminus \bigcup_{j \in \mathbb{Z}} \overline{\Theta_j} \tag{2.3}$$

is a domain with a Lipschitz boundary $\partial\Omega$ and divide $\partial\Omega$ into two parts, the submerged part Σ and the free surface Γ, i.e.

$$\Sigma = \{(x, y, z) \in \partial\Omega : z < 0\}, \qquad \Gamma = \partial\Omega \setminus \overline{\Sigma}.$$

Making all the usual assumptions for the linear water-wave theory to be valid (see *e.g.* Kuznetsov *et al.* 2002 and Kundu & Cohen 2004) and assuming, in particular, that the irrotational, small-amplitude wave motion is time-harmonic with the frequency $\omega > 0$, we are led to the following spectral problem for the complex velocity potential $\Phi = \Phi(x, y, z)$:

$$\left.\begin{array}{lll} \Delta\Phi = 0 & \text{in} & \Omega, \\ \partial_\nu \Phi = 0 & \text{on} & \Sigma, \qquad \partial_z \Phi = \lambda\Phi \quad \text{on} \quad \Gamma. \end{array}\right\} \tag{2.4}$$

Here Δ is the Laplace operator, $\lambda = g^{-1}\omega^2$ a spectral parameter and $g > 0$ the acceleration due to gravity. Note that the Lipschitz property of $\partial\Omega$ guarantees that the normal vector ν is defined almost everywhere on Σ and that the wave phenomena cannot occur in a finite region (cf. Nazarov & Taskinen 2008, 2010). Also observe that the normal derivative ∂_ν coincides with $\partial_z = \partial/\partial z$ at the free surface Γ. Since Ω is a domain, in particular, a connected set, the obstacles cannot form an infinite barrier extending in the y-direction and throughout the depth.

Our interest is to prove the existence of guided waves or trapped modes, that is, wave-like solutions to problem (2.4) that are propagating along (guided by) the obstacles but confined to (trapped in) their vicinity. Given the l-periodicity of the family of obstacles, it is thus natural to assume (see *e.g.* Wilcox 1984) that the velocity potential can be written as

$$\Phi(x, y + lj, z) = \mathrm{e}^{\mathrm{i}\beta l j}\Phi(x, y, z), \quad j \in \mathbb{Z},$$

where $\beta \in \mathbb{R}$. It then suffices to consider the following problem in a periodicity cell $\varpi = \Pi \setminus \overline{\Theta}$ (see figure 1*b*) and the cross-sectional representation of the same cell in figure 2(*b*):

$$\left.\begin{array}{lll} -\Delta\varphi = 0 & \text{in} & \varpi, \\ \partial_\nu \varphi = 0 & \text{on} & \sigma, \qquad \partial_z \varphi = \lambda\varphi \quad \text{on} \quad \gamma, \end{array}\right\} \tag{2.5a}$$

$$\varphi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\,\varphi|_{y=0}, \qquad \partial_y\varphi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\,\partial_y\varphi|_{y=0}, \tag{2.5b}$$

where σ and γ are the parts of the surfaces Σ and Γ lying on $\partial\varpi$ and $\beta \in (-\pi/l, \pi/l]$ (cf. Wilcox 1984). The quasi-periodicity conditions (2.5*b*) guarantee that the velocity potential Φ, defined in $\varpi_j = \Pi_j \setminus \overline{\Theta}_j$, $j \in \mathbb{Z}$, by the formula

$$\Phi(x, y, z) = \mathrm{e}^{\mathrm{i}\beta l j}\,\varphi(x, y - lj, z), \tag{2.6}$$

satisfies problem (2.4) and changes smoothly from cell to cell.

To simplify the presentation, we assume that $\beta \in [0, \pi/l]$. We emphasize the irrelevance of the sign of β to our main conclusions; see the sufficient conditions (4.3) and (8.3): neither is influenced by the change $\beta \to -\beta$. The case $\beta = 0$ corresponds to pure periodicity conditions in (2.5*b*) and needs to be treated separately since any trapped mode is necessarily embedded in the continuous spectrum. These modes, provided they exist, correspond to standing waves, as do the trapped modes found when $\beta = \pi/l$.

3. Spectrum of the problem in the periodicity cell

Let $(\cdot,\cdot)_\varpi$ denote the natural scalar product in the Lebesgue space $L_2(\varpi)$ and $(\cdot,\cdot)_\gamma$ the scalar product in $L_2(\gamma)$. Moreover, let $\mathscr{H}_\beta$ be the subspace of functions in $H^1(\varpi)$ that satisfy the Dirichlet quasi-periodicity condition in (2.5*b*), i.e.

$$\mathscr{H}_\beta = \{\psi \in H^1(\varpi) \ : \ \psi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\,\psi|_{y=0}\}.$$

We will consider problem (2.5*a*)–(2.5*b*) in the following variational form (cf. Ladyzhenskaya 1985):

$$(\nabla\varphi, \nabla\psi)_\varpi = \lambda\,(\varphi,\psi)_\gamma, \quad \psi \in \mathscr{H}_\beta. \tag{3.1}$$

For $\beta \in (0, \pi/l]$, the left-hand side of (3.1) defines a scalar product $\langle\cdot,\cdot\rangle_\beta$ in the Hilbert space $\mathscr{H}_\beta$. However, note that although the embedding $\mathscr{H}_\beta \subset L_2(\gamma)$ is continuous (cf. Ladyzhenskaya 1985), it cannot be compact since the boundary γ is unbounded. This means that the spectrum of problem (3.1) is not purely discrete (cf. Birman & Solomjak 1987, Theorem 9.2.1).

The periodicity cell ϖ can also be understood as a domain with two cylindrical outlets to infinity. Therefore, if for some $\lambda \in \mathbb{C}$ problem (3.1) admits a solution with finite Dirichlet integral, then this solution must decay exponentially as $x \to \pm\infty$ (cf. Nazarov & Plamenevsky 1994, Chapter 5). The corresponding solution Φ to problem (2.4) is thus localized in the vicinity of the family of obstacles (2.2), decays exponentially at infinity, has finite energy per unit cell of width l and is called a ‘guided wave’ or ‘trapped mode’ (cf. Ursell 1951, 1987).

3.1. *The model problem*

To analyse the spectrum of problem (3.1), we consider the following model problem in the periodicity cell without obstacles:

$$\left.\begin{aligned} -\Delta\phi &= 0 \quad \text{in} \quad \Pi, \\ \partial_z\phi|_{z=-h} &= 0, \quad \partial_z\phi|_{z=0} = \lambda\,\phi|_{z=0}, \\ \phi|_{y=l} &= \mathrm{e}^{\mathrm{i}\beta l}\,\phi|_{y=0}, \quad \partial_y\phi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\,\partial_y\phi|_{y=0}. \end{aligned}\right\} \tag{3.2}$$

The wave-like solutions to this problem are of the form

$$\phi(x,y,z) = A\,\cosh k(z+h)\,\mathrm{e}^{\mathrm{i}\beta y}\exp(\pm\mathrm{i}\sqrt{k^2-\beta^2}\,x), \qquad A \in \mathbb{C},$$

with $k > 0$ satisfying the dispersion relation

$$\lambda = k\,\tanh kh. \tag{3.3}$$

There is a threshold value at $k = \beta$ above which the solutions (waves) are propagating to infinity in the x-direction. The corresponding cutoff value

$$\lambda_\beta^\dagger = \beta\,\tanh\beta h \tag{3.4}$$

is the lower bound of the continuous spectrum of problem (3.2); note that $\lambda(k)$ is a monotonous function of k and that $\lambda_\beta^\dagger > 0$ for all $\beta > 0$. The solution corresponding to $\lambda_\beta^\dagger$, up to a multiplication by an arbitrary non-zero complex constant, is given by

$$\phi_\beta^\dagger(y,z) = \mathrm{e}^{\mathrm{i}\beta y}\,\cosh\beta(z+h). \tag{3.5}$$

FIGURE 3. The essential spectrum of the operator T_β.

In other words, recalling the general theory of elliptic boundary-value problems in domains with cylindrical outlets to infinity (see Kondratiev 1967, the monograph Nazarov & Plamenevsky 1994 and the review article Nazarov 2009*c*), the operator of problem (3.1), regarded as a mapping $\mathscr{H}_\beta \to \mathscr{H}_\beta^*$, loses its Fredholm property if and only if $\lambda \in \mathbb{C}$ belongs to the ray $[\lambda_\beta^\dagger, +\infty) \subset \mathbb{R}$. Moreover, the mapping is an isomorphism as long as λ does not belong to the spectrum, that is, it belongs neither to the continuous spectrum $[\lambda_\beta^\dagger, +\infty)$ nor to the discrete spectrum of problem (3.1). Besides, its discrete spectrum, if non-empty, lies on the segment $[0, \lambda_\beta^\dagger) \subset \mathbb{R}$. The trapped modes correspond to possible eigenvalues in the discrete spectrum. If $\beta = 0$ then $\lambda_0^\dagger = 0$ and the discrete spectrum is empty.

3.2. *Operator formulation*

We will establish the existence of eigenvalues in $[0, \lambda_\beta^\dagger)$ by making use of the spectral theory of self-adjoint operators in Hilbert space (cf. Birman & Solomjak 1987). Towards this end and following Nazarov (2008), we introduce a trace operator $T_\beta : \mathscr{H}_\beta \to L_2(\gamma)$ by

$$\langle T_\beta \varphi, \psi \rangle_\beta = (\varphi, \psi)_\gamma, \quad \varphi, \psi \in \mathscr{H}_\beta. \tag{3.6}$$

The linear operator T_β is positive, symmetric and continuous, and therefore self-adjoint; recall that $\mathscr{H}_\beta$ is continuously embedded into $L_2(\gamma)$. The spectral problem (3.1) is equivalent to

$$T_\beta \varphi = \mu \varphi \quad \text{in} \quad \mathscr{H}_\beta, \tag{3.7}$$

with $\mu = 1/\lambda$ denoting the new spectral parameter. It is clear that the operator T_β inherits from problem (3.1) the continuous spectrum $(0, \mu_\beta^\dagger]$, with $\mu_\beta^\dagger = (\lambda_\beta^\dagger)^{-1}$. Besides, $\mu = 0$ is an eigenvalue of infinite multiplicity and the associated eigenspace is composed of functions in $\mathscr{H}_\beta$ that vanish on γ. Hence, the essential spectrum of the operator T_β is $[0, \mu_\beta^\dagger]$. Note that the value $\mu = 0$ corresponds to $\lambda = \infty$, which, of course, does not influence the spectrum of problem (3.1).

4. A condition for the existence of trapped modes: the case $\beta > 0$

Trapped modes are associated with eigenvalues in the discrete spectrum of the self-adjoint operator T_β or, equivalently, with eigenvalues in the discrete spectrum of problem (3.1). First recall that the whole spectrum of T_β belongs to the segment $[0, \tau_\beta] \subset \mathbb{R}$ in the complex plane, with τ_β denoting the norm of the operator T_β (cf. Birman & Solomjak 1987, Theorem 10.2.1). Hence, if $\tau_\beta = \mu_\beta^\dagger$, the spectrum of T_β coincides with the essential spectrum and the discrete spectrum is empty. On the other hand, if

$$\tau_\beta > \mu_\beta^\dagger, \tag{4.1}$$

then the discrete spectrum is certainly non-empty since, in particular, $\mu_1 = \tau_\beta$ is an eigenvalue of T_β (see Birman & Solomjak 1987, Theorem 10.2.1 and figure 3), in fact the largest one. The associated eigenfunction $\varphi_1 \in \mathscr{H}_\beta$ satisfies (3.7) with $\mu = \mu_1$ and corresponds to a trapped mode in problem (3.1) with $\lambda = 1/\mu_1$.

Let us choose a trial function $\varphi^\epsilon \in \mathscr{H}_\beta$ such that

$$\varphi^\epsilon(x, y, z) = \mathrm{e}^{-\epsilon|x|}\phi^\dagger_\beta(y, z),$$

where $\epsilon \ll 1$ is a positive parameter and $\phi^\dagger_\beta$ is the solution corresponding to the cutoff value $\lambda^\dagger_\beta$; cf. (3.5). Recalling the definition of the operator norm

$$\tau_\beta = \sup_{\varphi\in\mathscr{H}_\beta\setminus\{0\}} \frac{\langle T_\beta\varphi, \varphi\rangle_\beta}{\langle\varphi, \varphi\rangle_\beta} = \sup_{\varphi\in\mathscr{H}_\beta\setminus\{0\}} \frac{\|\varphi; L_2(\gamma)\|^2}{\|\nabla\varphi; L_2(\varpi)\|^2},$$

we calculate

$$\begin{aligned}
\|\varphi^\epsilon; L_2(\gamma)\|^2 &= \int_{\mathbb{R}} \mathrm{e}^{-2\epsilon|x|}\,\mathrm{d}x \int_0^l |\phi^\dagger_\beta(y, 0)|^2\,\mathrm{d}y - \int_\theta (1 + O(\epsilon))|\phi^\dagger_\beta(y, 0)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&= \epsilon^{-1}\, l\,\big(\cosh\beta h\big)^2 - \big(\cosh\beta h\big)^2 \operatorname{meas}_2\theta + O(\epsilon), \\
\|\nabla\varphi^\epsilon; L_2(\varpi)\|^2 &= \|\nabla\varphi^\epsilon; L_2(\Pi)\|^2 - \|\nabla\varphi^\epsilon; L_2(\Theta)\|^2 \\
&= \epsilon^{-1}\, l\,\beta\,\sinh\beta h\,\cosh\beta h - \beta\, I_\beta(\Theta) + O(\epsilon).
\end{aligned}$$

Here, $\operatorname{meas}_2\theta$ denotes the cross-sectional area at $z = 0$ of the part(s) of the obstacle(s) piercing the free surface, i.e. $\overline{\theta} = \{(x, y) : (x, y, 0) \in \overline{\Theta}\}$ and

$$I_\beta(\Theta) = \beta \int_\Theta \cosh 2\beta(h + z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Moreover, we have taken into account that in any compact set we can write $\mathrm{e}^{-\epsilon|x|} = 1 + O(\epsilon)$ since $\epsilon \ll 1$.

Collecting these calculations, we conclude that

$$\begin{aligned}
\tau_\beta &\geqslant \frac{\|\varphi^\epsilon; L_2(\gamma)\|^2}{\|\nabla\varphi^\epsilon; L_2(\varpi)\|^2} \\
&\geqslant \frac{1}{\beta}\,\frac{\cosh\beta h}{\sinh\beta h}\,\frac{1 - \epsilon\, l^{-1}\operatorname{meas}_2\theta - c\,\epsilon^2}{1 - \epsilon\, l^{-1}(\sinh\beta h\cosh\beta h)^{-1}\, I_\beta(\Theta) + c\,\epsilon^2} \qquad (4.2) \\
&\geqslant \mu^\dagger_\beta + \epsilon\, l^{-1}\mu^\dagger_\beta\{(\sinh\beta h\cosh\beta h)^{-1}\, I_\beta(\Theta) - \operatorname{meas}_2\theta\} - C\epsilon^2,
\end{aligned}$$

where $C \in \mathbb{R}$ stands for some positive constant. If the term in the curly brackets above is positive, then, for small enough $\epsilon > 0$, the last lower bound in (4.2) exceeds the upper bound $\mu^\dagger_\beta$ of the continuous spectrum of the operator T_β, i.e. formula (4.1) is valid. In other words,

$$\frac{2\beta}{\sinh 2\beta h}\int_\Theta \cosh 2\beta(h + z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z > \operatorname{meas}_2\theta \qquad (4.3)$$

is a sufficient condition for the existence of a trapped mode.

5. A condition for the existence of trapped modes: the case $\beta = 0$

When $\beta = 0$, the reasoning of the previous section does not work since $\lambda^\dagger_\beta = 0$ and the discrete spectrum of problem (3.1) is empty. However, we may look for trapped modes embedded in the continuous spectrum following the approach proposed in Evans *et al.* (1994). Towards this aim, we assume that the member Θ_j in the array

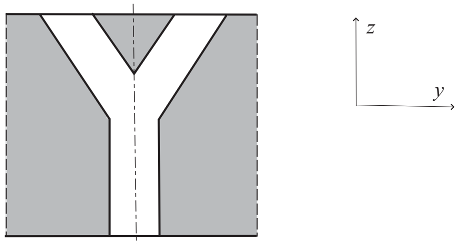

FIGURE 4. An obstacle satisfying the symmetry assumption.

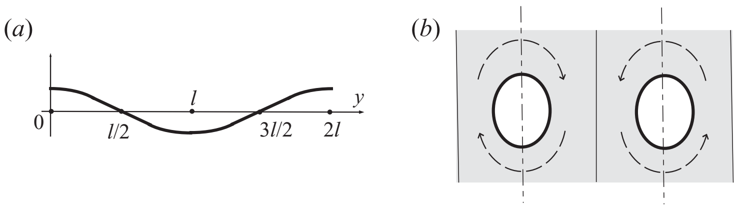

FIGURE 5. The extended solution (*a*) and the doubled periodicity cell (*b*).

$\{\Theta_j\}_{j\in\mathbb{Z}}$ of identical obstacles is symmetric with respect to the plane $\{y=lj/2\}$. The model obstacle Θ is thus symmetric with respect to the plane $\{y=l/2\}$, i.e.

$$\Theta = \{(x, y, z) : (x, -y + l/2, z) \in \Theta\}. \tag{5.1}$$

See the example in figure 4. We may now consider, instead of problem (2.5*a*)–(2.5*b*), the following spectral boundary-value problem with artificial Neumann and Dirichlet boundary conditions:

$$\left.\begin{aligned} -\Delta\varphi_\bullet &= 0 \quad \text{in} \quad \varpi_\bullet, \\ \partial_\nu\varphi_\bullet &= 0 \quad \text{on} \quad \sigma_\bullet, \quad \partial_z\varphi_\bullet = \lambda_\bullet\varphi_\bullet \quad \text{on} \quad \gamma_\bullet, \end{aligned}\right\} \tag{5.2a}$$

$$\partial_y\varphi_\bullet|_{y=0} = 0, \qquad \varphi_\bullet|_{y=l/2} = 0, \tag{5.2b}$$

where $\varpi_\bullet = \{(x, y, z) \in \varpi \; : \; y \in (0, l/2)\}$ denotes the left-half of the periodicity cell ϖ; $\sigma_\bullet$ and $\gamma_\bullet$ can be defined similarly. The boundary condition (5.2*b*) still guarantees that any solution $\{\lambda_\bullet, \varphi_\bullet\} \in \mathbb{R} \times H^1(\varpi_\bullet)$ can be extended smoothly through the lateral boundaries, now by an odd extension at $\{y=l/2\}$ and by an even one at $\{y=0\}$. The symmetry assumptions on the family of obstacles with respect to the planes at $\{y=0\}$ and $\{y=l/2\}$ imply that the extension satisfies (5.2*a*) in the enlarged sets. Moreover, it is easy to see that the extension φ of $\varphi_\bullet$ from $\varpi_\bullet$ into ϖ is anti-periodic in the periodicity cell ϖ, that is, it meets condition (2.5*b*) with $\beta = \pi/l$ (see also figure 5*a*). Therefore, $\{\lambda_\bullet, \varphi\} \in \mathbb{R} \times H^1_\pi(\varpi)$ is a solution, after an even extension at $\{y=l\}$, to problem (2.5*a*)–(2.5*b*), with $\beta=0$, in the periodicity cell of width $2l$ containing two (groups of) identical obstacles. We refer to Porter & Evans (2005) for a similar consideration.

In other words, any trapped mode found for problem (5.2*a*) is passed on to problem (2.5*a*)–(2.5*b*), written, with $\beta=0$, in the doubled periodicity cell of width $2l$ and provided, of course, that the symmetry requirement (5.1) is satisfied. The fact that the cell ϖ has been enlarged makes no difference since we could have assumed from the very beginning that the period is $2l$.

Solving the model spectral problem

$$\left.\begin{aligned} -\Delta\phi_\bullet &= 0 \quad \text{in } \Pi_\bullet, \\ \partial_z\phi_\bullet|_{z=-h} &= 0, \quad \partial_z\phi_\bullet|_{z=0} = \lambda\,\phi_\bullet|_{z=0}, \\ \partial_y\phi_\bullet|_{y=0} &= 0, \quad \phi_\bullet|_{y=l/2} = 0, \end{aligned}\right\} \tag{5.3}$$

where $\Pi_\bullet = \{(x, y, z) \in \Pi : y \in (0, l/2)\}$, we obtain the eigenfunctions

$$\phi_\bullet(x, y, z) = A \cosh k(z+h) \cos\frac{\pi}{l} y \exp\left(\pm \mathrm{i}\sqrt{k^2 - \left(\frac{\pi}{l}\right)^2} x\right), \qquad A \in \mathbb{C},$$

and the eigenvalues $\lambda_\bullet = k \tanh(kh)$. Hence, the first (cutoff) eigenvalue is

$$\lambda_\bullet^\dagger = \frac{\pi}{l} \tanh\frac{\pi}{l} h,$$

and the corresponding eigenfunction, up to a multiplication by an arbitrary non-zero constant, is

$$\phi_\bullet^\dagger(y, z) = \cos\frac{\pi}{l}\, y \cosh\frac{\pi}{l}(z+h).$$

We thus conclude that the continuous spectrum of problem (5.2*a*), formulated due to the symmetry assumption (5.1) in the new periodicity cell $\varpi_\bullet$, has a positive lower bound.

We can write problem (5.2*a*) in the form

$$(\nabla\varphi_\bullet, \nabla\psi_\bullet)_{\varpi_\bullet} = \lambda_\bullet\,(\varphi_\bullet, \psi_\bullet)_{\gamma_\bullet}, \quad \psi_\bullet \in \mathscr{H}_\bullet, \tag{5.4}$$

where $\mathscr{H}_\bullet$ denotes the Hilbert space of functions in $H^1(\varpi_\bullet)$ vanishing at $y = l/2$. Due to the Dirichlet boundary conditions, the bi-linear form $(\nabla\,\cdot, \nabla\,\cdot)_{\varpi_\bullet}$ defines a scalar product $\langle\cdot, \cdot\rangle_\bullet$ in $\mathscr{H}_\bullet$, and we can thus define a trace operator $T_\bullet$ as

$$\langle T_\bullet\varphi, \psi\rangle_\bullet = (\varphi, \psi)_{\gamma_\bullet}, \quad \varphi, \psi \in \mathscr{H}_\bullet, \tag{5.5}$$

denote its norm by $\tau_\bullet$ and analyse, following the method of the previous section, under which conditions the inequality $\tau_\bullet > (\lambda_\bullet^\dagger)^{-1}$ holds.

Defining the trial function $\varphi_\bullet^\epsilon(x, y, z) = \mathrm{e}^{-\epsilon|x|}\phi_\bullet^\dagger(y, z)$, we obtain

$$\begin{aligned} \tau_\bullet &\geqslant \frac{\|\varphi_\bullet^\epsilon; L_2(\gamma_\bullet)\|^2}{\|\nabla\varphi_\bullet^\epsilon; L_2(\varpi_\bullet)\|^2} \\ &\geqslant \frac{l}{\pi}\,\frac{\cosh\frac{\pi}{l}h}{\sinh\frac{\pi}{l}h}\;\frac{1 - 4\epsilon\, l^{-1} I_\bullet(\theta_\bullet) - c\,\epsilon^2}{1 - 4\epsilon\,\pi\, l^{-2}\left(\sinh\frac{\pi}{l}h\cosh\frac{\pi}{l}h\right)^{-1} I_\bullet(\Theta_\bullet) + c\,\epsilon^2} \\ &\geqslant (\lambda_\bullet^\dagger)^{-1} + 4\,\epsilon\, l^{-1}(\lambda_\bullet^\dagger)^{-1}\left\{\pi\, l^{-1}\left(\sinh\frac{\pi}{l}h\,\cosh\frac{\pi}{l}h\right)^{-1} I_\bullet(\Theta_\bullet) - I_\bullet(\theta_\bullet)\right\} - C\epsilon^2, \end{aligned} \tag{5.6}$$

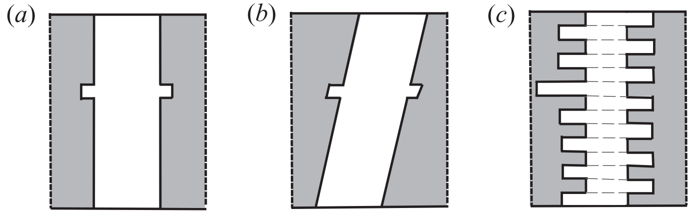

FIGURE 6. Cavalieri's principle.

where c and C are some positive constants (independent of ϵ) and

$$I_\bullet(\Theta_\bullet) = \int_{\Theta_\bullet} \left(\left(\cos \frac{\pi}{l} y \right)^2 \left(\sinh \frac{\pi}{l}(z+h) \right)^2 + \left(\sin \frac{\pi}{l} y \right)^2 \left(\cosh \frac{\pi}{l}(z+h) \right)^2 \right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

$$I_\bullet(\theta_\bullet) = \int_{\theta_\bullet} \left(\cos \frac{\pi}{l} y \right)^2 \mathrm{d}x\, \mathrm{d}y.$$

A sufficient condition for the existence of a trapped mode in problem (5.2*a*) is thus

$$\int_{\Theta} \left(\left(\cos \frac{\pi}{l} y \right)^2 \left(\sinh \frac{\pi}{l}(z+h) \right)^2 + \left(\sin \frac{\pi}{l} y \right)^2 \left(\cosh \frac{\pi}{l}(z+h) \right)^2 \right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$
$$> \frac{l}{\pi} \sinh \frac{\pi}{l} h \cosh \frac{\pi}{l} h \int_{\theta} \left(\cos \frac{\pi}{l} y \right)^2 \mathrm{d}x\, \mathrm{d}y, \quad (5.7)$$

where, in view of the symmetry assumption (5.1), we have written the condition in terms of the entire sets θ and Θ.

Remark. Note that the eigenvalue $\lambda_\bullet = \tau_\bullet^{-1}$ corresponding to a trapped mode and belonging to the discrete spectrum of problem (5.2*a*) is embedded into the continuous spectrum of problem (2.5*a*)–(2.5*b*), with $\beta = 0$, formulated in a periodicity cell of width $2l$ containing two identical obstacles. If the obstacles are circular columns extending throughout the water depth, even and odd extensions of the solution lead to a standing wave motion around the columns depicted in figure 5.

6. Examples: the case $\beta > 0$

6.1. *Submerged obstacles*

If the periodic family of obstacles is submerged, then $\mathrm{meas}_2\, \theta = 0$ and condition (4.3) is trivially true. Therefore, as is well known, a trapped mode exists. In particular, any submerged periodic mountain ridge produces a trapped edge wave.

6.2. *Cavalieri's principle*

Assume that condition (4.3) holds for Θ and that $\Theta^\sharp$ is another obstacle, different from Θ, such that the horizontal cross-sections of Θ and $\Theta^\sharp$ coincide at each depth (cf. figure 6). Cavalieri's principle, stating that two objects situated between two parallel planes have equal volumes if every plane parallel to these two planes intersects both objects in cross-sections of equal area, implies that the values of the volume integral, being independent of x and y, and the surface measure in condition (4.3) are the same for the two obstacles and, hence, the array of obstacles $\Theta^\sharp$ also produces a trapped mode.

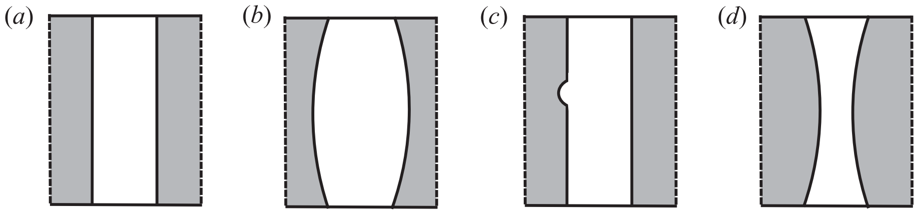

FIGURE 7. Surface-piercing obstacles.

6.3. *Surface-piercing columns*

If the surface-piercing column has a uniform cross-section (cf. figure 7a), the integral on the left-hand side of inequality (4.3) reduces to $\mathrm{meas}_2\,\theta$ (see formula (6.1) in § 6.4). This means that any bulged surface-piercing column (cf. figure 7b and c) produces a trapped mode since by enlarging the submerged part we increase the integral on the left-hand side of (4.3). Our technique, however, does not lead to any conclusion when the columns are bent as in figure 7(d).

In the next section, we will establish, in particular, the well-known result (cf. Sukhinin 1998; Linton & McIver 2002; Kamotskii & Nazarov 2003) stating that a periodic array of vertical surface-piercing columns with uniform cross-section (cf. figure 7a) produces a trapped edge wave.

6.4. *Surface-piercing columns with uniform cross-section*

If the obstacles extend uniformly throughout the water depth, that is, if they are vertical columns with uniform cross-section (cf. figure 2), then it is easy to see that

$$2\,\beta\,(\sinh 2\beta h)^{-1}\int_{\Theta}\cosh 2\beta(h+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\mathrm{meas}_2\,\theta. \tag{6.1}$$

Hence, our sufficient condition is not met although it is known that trapped edge waves exist. In fact, this is a borderline case in which we need to follow the second part of the variational method outlined in Kamotskii & Nazarov (2003) and redefine our trial function in order to prove the existence of a trapped mode. The advantage of the approach in Kamotskii & Nazarov (2003) (see also Nazarov & Videman 2009) is that, in contrast to Sukhinin (1998) and Linton & McIver (2002), the trial functions are not geometry-dependent and the method can thus be easily extended to the three-dimensional situation.

We choose $\varphi^\epsilon\in\mathscr{H}_\beta$ such that

$$\varphi^\epsilon(x,y,z)=\mathrm{e}^{-\epsilon|x|}\phi_\beta^\dagger(y,z)+\sqrt{\epsilon}\,\Psi(x,y,z),$$

where Ψ is a complex smooth function in Π with compact support in the neighbourhood of column Θ. As in § 4, we calculate

$$\langle T_\beta\varphi_\epsilon,\varphi_\epsilon\rangle_\beta=\frac{1}{\epsilon}A-P+2\sqrt{\epsilon}\,\langle T_\beta\phi_\beta^\dagger,\Psi\rangle_\beta+O(\epsilon),$$

$$\langle\varphi_\epsilon,\varphi_\epsilon\rangle_\beta=\frac{1}{\epsilon}B-V+2\sqrt{\epsilon}\,\langle\phi_\beta^\dagger,\Psi\rangle_\beta+O(\epsilon),$$

where

$$A=l\,(\cosh\beta h)^2,\qquad B=\beta l\,\sinh\beta h\,\cosh\beta h,$$

$$P=(\cosh\beta h)^2\,\mathrm{meas}_2\theta,\quad V=\beta^2\int_{\Theta}\cosh 2\beta(h+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

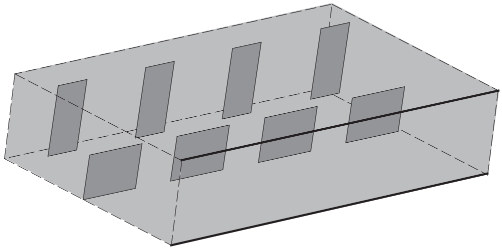

FIGURE 8. Arrays of vertical screens.

In order to show that there exists a trapped mode corresponding to an eigenvalue in the discrete spectrum of operator T_β, assume, for the sake of contradiction, that the discrete spectrum is empty, that is, $\tau_\beta = (\lambda_\beta^\dagger)^{-1}$. Hence, by definition of the operator norm, the inequality

$$\langle \varphi_\epsilon, \varphi_\epsilon \rangle_\beta \geqslant \lambda_\beta^\dagger \, \langle T_\beta \varphi_\epsilon, \varphi_\epsilon \rangle_\beta \tag{6.2}$$

must hold. Therefore,

$$\begin{aligned} &\langle \varphi_\epsilon, \varphi_\epsilon \rangle_\beta - \lambda_\beta^\dagger \, \langle T_\beta \varphi_\epsilon, \varphi_\epsilon \rangle_\beta \\ &\quad = \frac{1}{\epsilon} (B - \lambda_\beta^\dagger \, A) + \lambda_\beta^\dagger \, P - V + 2\sqrt{\epsilon}\, (\langle \phi_\beta^\dagger, \Psi \rangle_\beta - \lambda_\beta^\dagger \, \langle T_\beta \phi_\beta^\dagger, \Psi \rangle_\beta) + O(\epsilon) \\ &\quad = 2\sqrt{\epsilon}\ \mathrm{Re}\, (\langle \phi_\beta^\dagger, \Psi \rangle_\beta - \lambda_\beta^\dagger \, \langle T_\beta \phi_\beta^\dagger, \Psi \rangle_\beta) + C\epsilon \geqslant 0, \end{aligned}$$

where $C > 0$ is a constant independent of ϵ and where we have taken into account that $\lambda_\beta^\dagger \, A = B$ and $V = \lambda_\beta^\dagger \, P$, in view of (6.1). Integrating the term $\langle \phi_\beta^\dagger, \Psi \rangle_\beta$ in parts over ϖ, observing that all integrals are convergent since Ψ has compact support in ϖ and recalling that $\phi_\beta^\dagger$ satisfies the model problem (3.2), with $\lambda = \lambda_\beta^\dagger$, we obtain

$$2\sqrt{\epsilon}\ \mathrm{Re} \int_{S_\Theta} \partial_n \phi_\beta^\dagger \, \overline{\Psi} \, \mathrm{d}s + C\epsilon \geqslant 0,$$

where S_Θ denotes the lateral surface of the column Θ. Now, if $\partial_n \phi_\beta^\dagger$ is not identically zero almost everywhere, then we can choose Ψ in such a way that

$$\mathrm{Re} \int_{S_\Theta} \partial_n \phi_\beta^\dagger \, \overline{\Psi} \, \mathrm{d}s < 0.$$

Therefore, for small enough $\epsilon > 0$,

$$\langle \varphi_\epsilon, \varphi_\epsilon \rangle_\beta - \lambda_\beta^\dagger \, \langle T_\beta \varphi_\epsilon, \varphi_\epsilon \rangle_\beta < 0, \tag{6.3}$$

which contradicts (6.2). Consequently, the discrete spectrum of T_β is non-empty and a trapped mode exists.

The only situation where we cannot choose $\epsilon > 0$ to satisfy (6.3) is when $\partial_n \phi_\beta^\dagger$ vanishes identically along S_Θ. This situation corresponds to a vertical screen, or a union of vertical screens, aligned along the y-axis (cf. figure 8). In this case, the discrete spectrum of T_β is actually empty, as can be shown by the Friedrichs inequality:

$$\int_0^l |\phi(y,0)|^2 \, \mathrm{d}y \leqslant (\lambda_\beta^\dagger)^{-1} \int_0^l \int_{-h}^0 \left(\left| \frac{\partial \phi}{\partial y}(y,z) \right|^2 + \left| \frac{\partial \phi}{\partial z}(y,z) \right|^2 \right) \mathrm{d}y \, \mathrm{d}z. \tag{6.4}$$

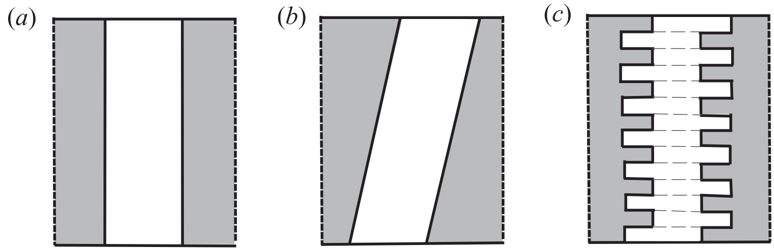

FIGURE 9. Cavalieri's principle for uniform columns.

Inequality (6.4) expresses the fact that $\lambda_\beta^\dagger > 0$ is the first (smallest) eigenvalue of the model problem (3.2) considered at any cross-section $\{x = \text{const.}\}$ of the periodicity cell Π. Given that the screen has zero thickness in the x-direction, we obtain, after integrating (6.4) in x,

$$\mu_\beta^\dagger \geqslant \frac{\displaystyle\int_\gamma |\phi|^2 \, \mathrm{d}y \, \mathrm{d}x}{\displaystyle\int_\varpi |\nabla\phi|^2 \, \mathrm{d}x \, \mathrm{d}y \, \, z} = \frac{\langle T_\beta \phi, \phi\rangle_\beta}{\langle \phi, \phi\rangle_\beta} \qquad \forall\, \phi \in \mathscr{H}_\beta \setminus \{0\}.$$

Consequently, the discrete spectrum of T_β is empty. Note that a trapped edge wave also exists when the columns are leaning, or of some other form with a uniform horizontal cross-section (cf. figure 9), since due to Cavalieri's principle, (6.1) holds and the previous reasoning is valid (see also § 6.2).

6.5. *Surface-piercing obstacles*

Taking (formally) the limit when $\beta \to 0$ in the term on the left-hand side of (4.3), one obtains the following condition from (4.3):

$$h^{-1}\mathrm{meas}_3\, \Theta > \mathrm{meas}_2\, \theta,$$

with $\mathrm{meas}_3\Theta$ denoting the volume of the obstacle Θ. Hence, if the mean area of the horizontal cross-sections of the obstacle Θ is larger than its cross-sectional area at the free surface, then a trapped mode exists for small enough $\beta > 0$. If the obstacles are columns extending throughout the depth, then from the previous example it follows that the same conclusion is valid for any $\beta > 0$.

6.6. *Comparison principles*

Let us assume that the array of obstacles Θ produces a trapped mode corresponding to an eigenvalue $\mu_1 = \tau_\beta > \mu_\beta^\dagger$ in the discrete spectrum of T_β and also consider another periodic array of (larger) obstacles $\Theta^\sharp$ such that

$$\Theta \subset \Theta^\sharp \subset \Pi, \qquad \Theta^\sharp \neq \Theta, \qquad \overline{\theta} = \overline{\theta^\sharp}, \qquad \gamma = \gamma^\sharp. \tag{6.5}$$

Note that the obstacles $\Theta^\sharp$ and Θ pierce the free surface at the same place. We assume that $\varpi^\sharp = \Pi \setminus \overline{\Theta^\sharp}$ is a Lipschitz domain; consider problem (2.5*a*)–(2.5*b*) in the periodicity cell $\varpi^\sharp$ and define a Hilbert space $\mathscr{H}_\beta^\sharp$ consisting of functions in $H^1(\varpi^\sharp)$ that satisfy the first quasi-periodicity condition in (2.5*b*). Moreover, analogously to (3.6), we introduce in $\mathscr{H}_\beta^\sharp$ an operator $T_\beta^\sharp$ associated with the spectral problem in $\varpi^\sharp$ by

$$\langle T_\beta^\sharp \varphi, \psi\rangle_\beta^\sharp = (\varphi, \psi)_{\gamma^\sharp}, \qquad \varphi, \psi \in \mathscr{H}_\beta^\sharp,$$

where $\langle \cdot, \cdot\rangle_\beta^\sharp$ and $(\cdot, \cdot)_{\gamma^\sharp}$ denote the scalar products in $\mathscr{H}_\beta^\sharp$ and $L_2(\gamma^\sharp)$, respectively.

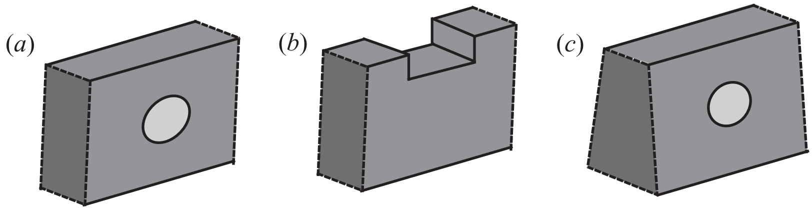

FIGURE 10. Barriers with apertures.

Let $\varphi_1 \in \mathscr{H}_\beta \subset \mathscr{H}^\sharp_\beta$ be the trapped-mode solution associated with the eigenvalue μ_1. Estimating the norm of the operator $T^\sharp_\beta$, we obtain

$$\tau^\sharp_\beta = \sup_{\varphi \in \mathscr{H}^\sharp_\beta \setminus \{0\}} \frac{\langle T^\sharp_\beta \varphi, \varphi \rangle^\sharp_\beta}{\langle \varphi, \varphi \rangle^\sharp_\beta} = \sup_{\varphi \in \mathscr{H}^\sharp_\beta \setminus \{0\}} \frac{(\varphi, \varphi)_{\gamma^\sharp}}{(\nabla \varphi, \nabla \varphi)_{\varpi^\sharp}}$$
$$\geqslant \frac{(\varphi_1, \varphi_1)_{\gamma^\sharp}}{(\nabla \varphi_1, \nabla \varphi_1)_{\varpi^\sharp}} > \frac{(\varphi_1, \varphi_1)_{\gamma}}{(\nabla \varphi_1, \nabla \varphi_1)_{\varpi}} = \mu_1 > \mu^\dagger_\beta, \quad (6.6)$$

since the harmonic function φ_1 cannot vanish in the non-empty open set $\varpi \setminus \overline{\varpi^\sharp}$. Therefore, given that the fluid domains ϖ and $\varpi^\sharp$ differ only over a compact set, the essential spectra of the operators $T^\sharp_\beta$ and T_β coincide and, consequently, (6.6) implies that the discrete spectrum of $T^\sharp_\beta$ is also non-empty. Moreover, the max–min principle (cf. Birman & Solomjak 1987, Theorem 10.2.2) shows that the total multiplicity of the discrete spectrum of $T^\sharp_\beta$ is not less than the total multiplicity of the discrete spectrum of T_β.

These types of results, known as comparison principles, have frequently been exploited in proving the existence of trapped modes (see e.g. Jones 1953 and Ursell 1987). However, the argument presented here (see also Nazarov 2009*a*,*b* and Nazarov & Videman 2009) is much simpler than in the earlier papers.

6.7. *Barriers with apertures*

If the barrier is a vertical wall and the apertures are slots at the surface (see figure 10*b* for the part of the wall inside the prismatic periodicity cell), then trapped edge waves always exist, by comparison with the impermeable vertical wall which leads to equality (6.1). In fact, if the vertical extension of the slot is $d > 0$, it is easy to see that the sufficient condition (4.3) reduces to

$$1 > 1 - \frac{\sinh 2\beta(h-d)}{\sinh 2\beta h},$$

which is true for all $d \in (0, h)$. If the apertures are totally under water and the barrier is a vertical wall (see figure 10*a*), we cannot conclude anything because (4.3) is clearly violated. On the other hand, if the barrier is dam-like (figure 10*c*) and the width of the base of the dam is sufficiently wide that its weighted volume exceeds the weighted volume of the vertical wall, then condition (4.3) is again satisfied and a trapped mode exists.

6.8. *Non-necessity of the sufficient condition*

To see that condition (4.3) is not necessary, consider the surface-piercing column Θ illustrated in figure 11, obtained by cutting out thin horizontal slices from a vertical rectangular column close to the surface and adding them just underneath. Observing

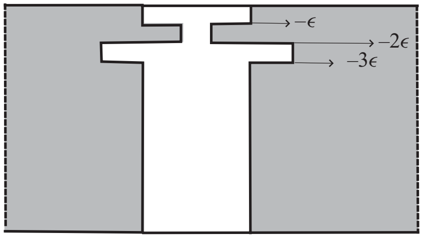

FIGURE 11. Obstacle producing trapped modes and violating the sufficient condition.

that the weight function $\cosh 2\beta(h+z)$ in the volume integral in (4.3) is an increasing function of $z \in (-h, 0)$ and recalling that the same volume integral coincides with the right-hand side of (4.3) if the column is vertical (cf. § 6.4), it is easy to see that

$$\frac{2\beta}{\sinh 2\beta h} \int_{\Theta} \cosh 2\beta(h+z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z < \mathrm{meas}_2\, \theta,$$

that is, our sufficient condition is not met. Then again, in Nazarov (2008) it was shown that, for any $\delta > 0$ and $N \in \mathbb{N}$, there exists $\epsilon_0 = \epsilon_0(\delta, N) > 0$ such that for $\epsilon \in (0, \epsilon_0(\delta, N))$, the spectral problem (2.5*a*)–(2.5*b*) admits at least N linearly independent trapped modes corresponding to N eigenvalues $\Lambda \in (0, \delta)$. Although the fluid domain and the surface-piercing obstacle considered in Nazarov (2008) were not as here, the conclusion, being based only on a local analysis and the max–min principle, remains valid.

7. Examples: the case $\beta = 0$

7.1. *Submerged obstacles*

In this case also, any periodic family of submerged obstacles produces a trapped mode since the right-hand side of condition (5.7) is zero and the left-hand side is positive.

7.2. *Surface-piercing obstacles*

Given that the weight functions in (5.7) also depend on y, the sufficient condition can also be valid when the obstacles are columns with uniform cross-section, but the conclusion referred to as the Cavalieri's principle (see § 6.2) no longer holds. If the surface-piercing obstacles are vertically uniform columns, the sufficient condition (5.7) reduces to

$$\left(\frac{h}{2} + \frac{l}{2} \sinh \frac{2\pi h}{l}\right) \int_{\theta} \cos \frac{2\pi}{l} y\, \mathrm{d}x\, \mathrm{d}y < 0. \tag{7.1}$$

This condition is satisfied with quite general cross-sections θ whenever the volume of the obstacle is mainly located in the area where $\cos(2\pi/l)y$ is negative, that is, in the interval $[l/4, 3l/4]$ around the centreline $\{y = l/2\}$. If the integral on the left-hand side of (7.1) happens to be zero, the existence of trapped modes can still be established following the reasoning of § 6.4.

7.3. *Comparison principles*

Assume that the family of obstacles $\{\Theta_j\}_{j\in\mathbb{Z}}$ produces a trapped edge wave, and let $\{\Theta_j^\sharp\}_{j\in\mathbb{Z}}$ be another array of identical obstacles such that $\Theta_j^\sharp$ is symmetric with

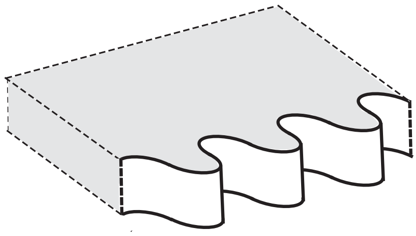

FIGURE 12. Periodic coastline with a vertical cliff face.

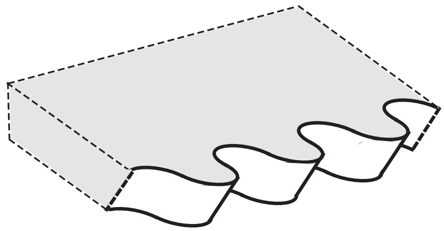

FIGURE 13. Periodically varying coastline.

respect to the plane $\{y = lj/2\}$. Moreover, assume that (6.5) holds. Then, reasoning as in § 6.6, we can also prove the existence of a trapped mode for the family $\{\Theta_j^\sharp\}_{j\in\mathbb{Z}}$ of larger obstacles.

8. Topographically guided edge waves

With some minor modifications, one can also establish a sufficient condition for the existence of linear water waves guided by a periodic topography. We will give two examples.

8.1. *Periodically varying coastline*

Let $H = H(y, z)$ be a smooth positive function defined in the strip $\mathbb{R} \times [-h, 0]$ and assume that H is l-periodic in the variable y. Moreover, let Ξ_H be a fluid layer defined by

$$\Xi_H = \{(x, y, z) : z \in (-h, 0),\ y \in \mathbb{R},\ x < -H(y, z)\}, \tag{8.1}$$

and let Γ_H denote the free surface at $z = 0$, Σ_H the flat bottom at $z = -h$ and Σ_H^0 the periodic lateral boundary at $x = -H(y, z)$ of the new fluid domain (see figure 12 for the case $H(y, z) = H(y)$ and figure 13 for the general case). Taking into account the l-periodicity of the lateral boundary, we can reduce the spectral problem for the velocity potential to the semi-infinite periodicity cell $\Pi_H = \{(x, y, z) \in \Xi_H : 0 < y < l\}$, namely

$$\left.\begin{aligned} -\Delta\varphi &= 0 \quad \text{in} \quad \Pi_H, \\ \partial_\nu\varphi &= 0 \quad \text{on} \quad \sigma_H \cup \sigma_H^0, \quad \partial_z\varphi = \lambda\varphi \quad \text{on} \quad \gamma_H, \end{aligned}\right\} \tag{8.2a}$$

$$\varphi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\,\varphi|_{y=0}, \quad \partial_y\varphi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\,\partial_y\varphi|_{y=0}, \tag{8.2b}$$

where σ_H, σ_H^0 and γ_H are the parts of Σ_H, Σ_H^0 and Γ_H intersecting $\partial\Pi_H$.

8.1.1. *A condition for the existence of trapped modes: the case $\beta > 0$*

Assuming that $\beta \in (0, \pi/l]$, we introduce a Hilbert space $\mathscr{H}_\beta$ consisting of functions in $H^1(\Pi_H)$ that satisfy the first quasi-periodicity condition in (8.2*b*), define in $\mathscr{H}_\beta$ a scalar product $\langle\varphi, \psi\rangle_\beta = (\nabla\varphi, \nabla\psi)_{\Pi_H}$ and introduce a trace operator T_β by the formula $\langle T_\beta\varphi, \psi\rangle_\beta = (\varphi, \psi)_{\gamma_H}$. The linear operator T_β is obviously self-adjoint and its essential spectrum coincides with the segment $[0, (\lambda_\beta^\dagger)^{-1}] \subset \mathbb{R}$, where $\lambda_\beta^\dagger > 0$ is the cutoff value given in (3.4).

Next, defining a trial function $\varphi^\epsilon(x, y, z) = \mathrm{e}^{-\epsilon x}\phi_\beta^\dagger(y, z)$, with $\epsilon \ll 1$, and where $\phi_\beta^\dagger$ is the solution from (3.5), we obtain

$$
\begin{aligned}
\|\varphi^\epsilon ; L_2(\gamma_H)\|^2 &= \int_{-\infty}^0 \mathrm{e}^{-2\epsilon x}\,\mathrm{d}x \int_0^l |\phi_\beta^\dagger(y,0)|^2\,\mathrm{d}y - \int_0^l \int_{-H(y,0)}^0 \mathrm{e}^{-2\epsilon x}\,\mathrm{d}x\; |\phi_\beta^\dagger(y,0)|^2\,\mathrm{d}y \\
&= (2\epsilon)^{-1}\, l\,(\cosh\beta h)^2 - (\cosh\beta h)^2 \int_0^l H(y,0)\,\mathrm{d}y + O(\epsilon), \\
\|\nabla\varphi^\epsilon ; L_2(\Pi_H)\|^2 &= \int_{-\infty}^0 \mathrm{e}^{-2\epsilon x}\,\mathrm{d}x \int_0^l \int_{-h}^0 |\nabla\phi_\beta^\dagger(y,z)|^2\,\mathrm{d}y\,\mathrm{d}z \\
&\quad - \int_0^l \int_{-h}^0 H(y,z)\,|\nabla\phi_\beta^\dagger(y,z)|^2\,\mathrm{d}y\,\mathrm{d}z + O(\epsilon) \\
&= (2\epsilon)^{-1}\, l\,\beta\, \sinh\beta h\, \cosh\beta h \\
&\quad - \beta^2 \int_0^l \int_{-h}^0 \cosh 2\beta(h+z)\, H(y,z)\,\mathrm{d}y\,\mathrm{d}z + O(\epsilon).
\end{aligned}
$$

Reasoning as in § 4, we arrive at the following condition that guarantees the existence of a trapped mode for problem (8.2*a*)–(8.2*b*) and, consequently, the existence of a trapped edge wave in problem (2.4) in the fluid domain (8.1):

$$
\int_0^l \int_{-h}^0 \cosh 2\beta(h+z)\, H(y,z)\,\mathrm{d}y\,\mathrm{d}z > \frac{\sinh 2\beta h}{2\beta} \int_0^l H(y,0)\,\mathrm{d}y. \tag{8.3}
$$

8.1.2. *A condition for the existence of trapped modes: the case $\beta = 0$*

If $\beta = 0$, we assume that $H = H(y, z)$ is, as well as l-periodic in y, an even function with respect to the variable $y - l/2$, i.e. $H((l/2) + y) = H((l/2) - y)$. Repeating the arguments of § 5, we obtain the following sufficient condition for the existence of a trapped mode:

$$
\begin{aligned}
&\int_0^l \int_{-h}^0 H(y,z)\left(\left(\cos\frac{\pi}{l}y\right)^2\left(\sinh\frac{\pi}{l}(z+h)\right)^2 + \left(\sin\frac{\pi}{l}y\right)^2\left(\cosh\frac{\pi}{l}(z+h)\right)^2\right)\mathrm{d}y\,\mathrm{d}z \\
&\qquad > \frac{l}{\pi}\sinh\frac{\pi}{l}h\,\cosh\frac{\pi}{l}h \int_0^l H(y,0)\left(\cos\frac{\pi}{l}y\right)^2\mathrm{d}y.
\end{aligned} \tag{8.4}
$$

8.1.3. *Examples*

Assume that $\beta \in (0, \pi/l]$. If the periodic lateral boundary is vertical, that is, if H is a function of y only, then the integrals on both sides in condition (8.3) coincide and no conclusions can be drawn on the existence of trapped modes. However, edge waves do exist, as was shown by Linton & McIver (2002) (see also Sukhinin 1998). Following Kamotskii & Nazarov (2003), we can redefine the trial function $\varphi^\epsilon \in \mathscr{H}_\beta$ as

$$
\varphi^\epsilon(x, y, z) = \mathrm{e}^{-\epsilon x}\phi_\beta^\dagger(y, z) + \epsilon^{1/2}\,\Psi(x, y, z),
$$

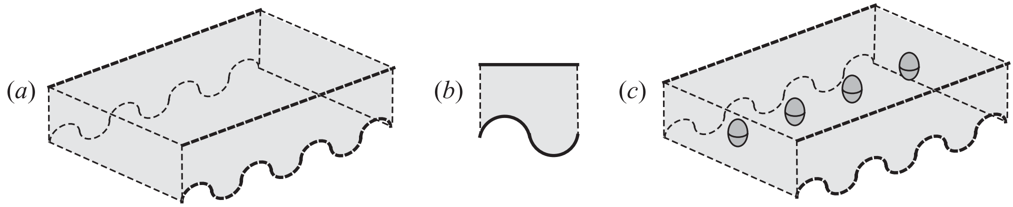

FIGURE 14. Periodically varying seabed (a,c), and the corresponding periodicity cell (b).

where Ψ is a smooth complex function in Π with compact support in the neighbourhood of the coast. This test function differs from the ones constructed in Sukhinin (1998) and Linton & McIver (2002), allows for Lipschitz boundaries and is independent of the particular structure of the boundary. Arguing as in § 6.4, it is easy to see that, provided we can choose Ψ in such a way that the integral

$$\mathrm{Re} \int_{-H(y)}^{0} \int_{0}^{l} \int_{-h}^{0} \partial_n \phi_\beta^\dagger \, \overline{\Psi} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

is negative, that is, provided $\partial_n \phi_\beta^\dagger$ does not vanish almost everywhere, then a trapped mode exists. Therefore, as long as the periodic coastline with a vertical cliff face is not straight, it produces a trapped edge wave. If $\beta = 0$ and the coastline is vertical, the sufficient condition (8.4) reduces to

$$\int_{0}^{l} H(y) \cos \frac{2\pi}{l} y \, \mathrm{d}y < 0 \, .$$

This inequality can be satisfied with some positive periodic functions $H = H(y)$ and violated with others. Note that only one Fourier coefficient of H needs to be negative to ensure the existence of a trapped mode. It is straightforward to establish the existence of edge waves along periodic coastlines with a sloping beach. In fact, the line integrals on the right-hand side of (4.3) and (5.7) remain the same as in the case of a coastline with a vertical cliff face but the integrals on the left-hand side are increased if the sea floor is sloping down towards the sea, that is, if $H(y, z) > H(y, 0)$, $z \in [-h, 0)$. This conclusion generalizes some of the results obtained by Bonnet-Ben Dhia & Joly (1993) along straight coastlines.

If the cliff face of the coast has some underwater caverns or is sloping backwards (see figure 13), our condition (8.3) is not satisfied, and the existence of trapped edge waves remains an open question. On the other hand, any combination of periodic underwater protrusions, with underwater caves such that the weighted volume of the protrusions is larger than the weighted volume of the caves, leads to a trapped mode.

8.2. *Periodically varying bottom topography*

A periodically varying ocean floor does not generate edge waves by itself. However, an infinite array of totally or partially submerged obstacles placed over a corrugated sea bottom does produce edge waves, as will be shown here. We will assume that the array of obstacles is aligned along the direction of periodicity of the corrugated sea bottom. In Nazarov (2009a), a somehow opposite but two-dimensional situation was considered, that is, an infinite horizontal cylinder placed over a periodically varying sea bottom with its axis perpendicular to the direction of periodicity of the bottom.

Assume that the bottom topography is defined by an l-periodic positive function $h = h(y)$ (cf. figure 14a) and that the infinite array of obstacles, aligned along the

y-axis (cf. figure 14c), is l-periodic. Consider the fluid layer $\Xi_h = \mathbb{R}^2 \times (-h(y), 0)$ and the periodicity cells $\Pi_h \subset \Xi_h$ (without obstacles) and $\varpi_h = \Pi_h \setminus \overline{\Theta}$. The model problem (3.2) cannot be solved analytically in Π_h, but one can still prove the existence of a positive lower bound for its continuous spectrum. In fact, making the Fourier transform in x, we obtain the following model problem in the cross-section S_h of Π_h with the (y, z)-plane (cf. figure 14b):

$$\left.\begin{array}{l} (-\Delta_{(y,z)} + \xi^2)\widehat{\phi} = 0 \quad \text{in} \quad S_h, \\ \partial_n \widehat{\phi}|_{z=-h(y)} = 0, \quad \partial_z \widehat{\phi}|_{z=0} = \lambda\, \widehat{\phi}|_{z=0}, \\ \widehat{\phi}|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\, \widehat{\phi}|_{y=0}, \quad \partial_y \widehat{\phi}|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\, \partial_y \widehat{\phi}|_{y=0}, \end{array}\right\} \tag{8.5}$$

where $\widehat{\phi} = \widehat{\phi}(\xi, y, z)$ is the Fourier transform of ϕ and ξ is the transform variable. It is clear that the smallest (cutoff) value of the continuous spectrum of the positive, self-adjoint operator of problem (8.5) corresponds to $\xi = 0$. Besides, this cutoff value $\lambda_1 = \lambda_1(\beta)$ is positive for all $\beta > 0$ and coincides with the cutoff value $\lambda_h^\dagger$ of the continuous spectrum of the corresponding problem in the periodicity cell ϖ_h (with obstacles).

We can now continue as before and define a variational formulation and a trace operator in the Hilbert space

$$\mathscr{H}_\beta = \{\psi \in H^1(\varpi_h) \,:\, \psi|_{y=l} = \mathrm{e}^{\mathrm{i}\beta l}\, \psi|_{y=0}\}.$$

This leads to the following sufficient condition, which guarantees the existence of trapped edge waves along a periodic array of obstacles placed over a periodic bottom:

$$\int_\Theta |\nabla \phi_h^\dagger(y, z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \lambda_h^\dagger \int_\theta |\phi_h^\dagger(y, 0)|^2 \,\mathrm{d}x\,\mathrm{d}y > 0. \tag{8.6}$$

Here $\phi_h^\dagger$ is the eigenfunction corresponding to the cutoff value $\lambda_h^\dagger$.

If $\beta = 0$, we assume that the obstacle Θ is symmetric with respect to the plane $\{y = l/2\}$ and consider a model problem in the left-half of the periodicity cell with the homogeneous Dirichlet boundary condition at $y = l/2$. We obtain a sufficient condition as in (8.6) with $\phi_h^\dagger$ and $\lambda_h^\dagger$ replaced with the eigenfunction/eigenvalue pair corresponding to the cutoff value of the continuous spectrum of the problem in the half of the periodicity cell.

8.2.1. *Examples*

Any periodic array of submerged obstacles, in particular an underwater mountain ridge, produces edge waves. A more interesting example is the situation when the obstacles are surface-piercing columns with uniform cross-section. Let us define a smooth, real-valued function F by

$$F(y) = \int_{I(y)} |\nabla \phi_h^\dagger(y, z)|^2 \,\mathrm{d}z - \lambda_h^\dagger\, |\phi_h^\dagger(y, 0)|^2,$$

where $I(y) = \{(y, z) \in S_h \,:\, -h(y) < z < 0\}$. Observing that $(\lambda_h^\dagger, \phi_h^\dagger)$ satisfies the model problem (8.5) (with $\xi = 0$), we obtain

$$\int_0^l F(y)\,\mathrm{d}y = 0.$$

Given that F cannot be constant, provided the bottom is not flat, it must be both positive and negative somewhere along the line $(0, l)$. Therefore, since the sufficient condition (8.6) can be written as

$$\int_{\theta} F(y)\,\mathrm{d}x\,\mathrm{d}y > 0,$$

we conclude that a trapped mode exists if the column is situated in a place where F is mostly positive. In the opposite case, that is, when F is mostly negative, condition (8.6) does not allow us to conclude anything about the existence of trapped modes. In general, the model problem (8.5) and the sufficient condition (8.6) must be investigated numerically.

9. Conclusions

We have examined the existence of trapped waves along infinite arrays of three-dimensional periodic structures. These types of trapped modes are known to exist when the geometry is either vertically uniform (an array of vertical columns or protrusions from a coastline with uniform cross-section over a flat bottom) or uniform (trivially periodic) in one horizontal direction (an infinitely long cylinder or a straight coastline with a sloping beach). All these cases can be reduced to two-dimensional problems and the corresponding trapped modes are often referred to as Rayleigh–Bloch surface waves or edge waves. Our variational approach, which is no more complicated in the three-dimensional case than in the two-dimensional case, leads to a simple geometrical condition guaranteeing the existence of a trapped mode. We have derived the condition for periodic arrays of identical obstacles of arbitrary shape, and for periodically varying coastlines and bottom topographies, and given a number of examples that lead to the existence of trapped edge waves. By pointing out cases where our sufficient condition does not tell us whether a trapped mode exists or not, we have come up with several open questions (see §§ 6.3, 6.7, 8.1.3 and 8.2.1).

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